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B. Sc. (Honrs) Part 2 paper 3

Subject:Mathematics

Title/Heading:Groups:Sub group

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Subgroups

Let n be a positive integer. If a is an element of a group G, written multiplicatively, we denote the product $aaa....a$ for n factors a by a^n. We let a^0 be the identity element. Also, a^{-n} denotes the product $a^{-1}a^{-1}a^{-1}...a^{-1}$ for n factors.

Definition

If G is a group, then the **order** $|G|$ of G is the number of elements in G.

Definition

If a subset H of a group G is closed under the binary operation and if H with the induced operation from G is itself a group, then H is a **subgroup** of G. We denote this by $H \leq G$ or $G \geq H$. Also, $H < G$ or $G > H$ means that $H \leq G$ but $H \neq G$.

Examples

1. If G is any group, then the subgroup consisting of G itself is the **improper subgroup** of G. All other subgroups of G are **proper subgroups**. The subgroup $\{e\}$ is the **trivial** subgroup of G. All other subgroups are **non-trivial**.

2. $\langle \mathbb{Z}, + \rangle < \langle \mathbb{R}, + \rangle$, but $\langle \mathbb{Q}^+, . \rangle$ is *not* a subgroup of $\langle \mathbb{R}, + \rangle$.

3. The n^{th} roots of unity in $\mathbb{C}$ form a subgroup U_n of the group $\mathbb{C}^*$ of non zero complex numbers under multiplication.

4. There are two different group structures of order 4. Consider the group table of $\mathbb{Z}_4$.

(In the problem 4 the operation is $+_4$, the addition modulo 4. $a+_4 b= r$, the reminder obtained when a+b is divided by 4)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

From the table, it is clear that the only proper subgroup of $\mathbb{Z}_4$is $\{0, 4\}$. Another group structure of order 4 is the group V, the **Klein 4-group**, which is described by the following table.

$V:$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Note that V has three proper nontrivial subgroups, $\{e, a\}$, $\{e, b\}$, and $\{e, c\}$.

Theorem

A subset H of a group G is a subgroup G if and only if (i). H is closed under the binary operation of G., (ii). the identity element e of G is in H, (iii). for all $a \in H$ it is true that $a^{-1} \in H$ also.

Theorem

Let G be a group and let $a \in G$. Then $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G and is the smallest subgroup of G that contains a, i.e., every subgroup containing a contains H.

Problem

Show that a non empty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

Solution.

Let H be a subgroup of G. Then for $a, b \in H$, we have $b^{-1} \in H$ and $ab^{-1} \in H$ because H must be closed under the induced operation. Conversely, suppose that H is nonempty and $ab^{-1} \in H$ for all $a, b \in H$. Let $a \in H$. Then taking $b = a$, we see that $aa^{-1} = e$ is in H. Taking $a = e$, and $b = a$, we see that $ea^{-1} = a^{-1} \in H$. Thus H contains the identity element and the inverse of each element. For closure, note that for $a, b \in H$, we also have $a, b^{-1} \in H$ and thus $a(b^{-1})^{-1} = ab \in H$. ■

Problem

Let G be a group and let $H_G = \{x \in G \mid xa = ax, \forall a \in G\}$. Show that H_G is an abelian subgroup of G. (H_G is called the **center** of G.)

Solution.

Clearly H_G is closed under the operation and $e \in H_G$. From $xa = ax$, we obtain $xax^{-1} = a$ and then $ax^{-1} = x^{-1}a$, showing that $x^{-1} \in H_G$, which is thus a subgroup. Let $a \in H_G$. Then $ag = ga$ for all $g \in G$; in particular,$ab = ba$ for all $b \in H_G$ because H_G is a subset of G . This shows that H_G is abelian. ■